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MATHEMATICAL MODELING OF DISCRETE NONCONSERVATIVE DYNAMIC SYSTEMS

J. A. WALKER

Department of Mechanical Engineering and Astronautical Sciences,
Northwestern University, Evanston, Illinois

Abstract—Many studies have shown that arbitrarily small differences between two nonconservative dynamic systems can result in completely different stability characteristics of the two systems. This can be interpreted as implying that mathematical modeling is of questionable value in the analysis and design of physical nonconservative systems. Using basic results from Liapunov stability theory, two rules for avoiding such infinite-sensitivity models are proposed for the mathematical modeling of discrete dynamic systems. Several general types of modeling error are considered, and these rules are shown to assure finite-sensitivity models.

INTRODUCTION

OVER the past decade there have appeared a wide variety of studies intended to demonstrate the possibly great sensitivity of nonconservative dynamic elastic systems to arbitrarily small parameter changes [1–12]. Specifically it has been very well documented that the addition of a small inertial term, or a small damping term, or a small time-delay to a mathematical model of a nonconservative system, may produce drastic changes in the stability properties of that model [1–12].

Since there is always some difference between a physical system (the modeled system) and the assumed mathematical model (the system model), the above studies cast considerable doubt on the value of studying the mathematical model of a nonconservative system. In fact many warnings have been issued to the effect that the system should be modeled very carefully. However very little has ever been said about how one goes about being careful.

If we symbolically denote the difference between the modeled system and the system model by g, then we clearly want a mathematical model to be one such that as $|g| \to 0$, the mathematical model becomes an arbitrarily accurate representation of the physical system, in the sense that the model behavior becomes an arbitrarily close approximation of the physical behavior; i.e. arbitrarily small modeling errors should lead to arbitrarily small errors in behavior description. All of the models considered above are ones which do not have this property and therefore they are unsatisfactory models. Alternatively we may say that the problems discussed above are infinite-sensitivity models and that only finite-sensitivity models can be termed satisfactory from an engineering standpoint.

Obviously even a finite-sensitivity model will predict the physical behavior incorrectly for $|g|$ sufficiently large, but this is both expected and unavoidable. Unlike an infinite-sensitivity model, however, a finite-sensitivity model gives a good description of the physical behavior provided $|g|$ is sufficiently small. The purpose of this work is to describe how one may determine that a given discrete model is a finite-sensitivity model.

MODELED SYSTEM

Here we will consider physical discrete systems which are instantaneous (no memory) and later we will make an extension to systems having memory. We assume there is an open region G of an n-dimensional Euclidean *state space* E^n such that, given any *initial state vector* $x_0 \in G$ at the initial time t_0, a unique *motion* $x(t; x_0) \in E^n$ for $t \in [t_0, \infty)$ results from the physical system.† Note that we assume that every $x \in G$ is a possible initial state. These assumptions imply that the physical system satisfies a mathematical rule of the form

$$\dot{x} = F(x, t), \qquad x(t_0) = x_0 \in G \subset E^n, x \in E^n, \qquad t \geq t_0 \tag{1}$$

which is valid for a given t_0 and any $x_0 \in G$. $F(x, t)$ is an n-vector function of present state and present time which is, of course, not known exactly.‡

We wish to study the stability of some given motion of this system. It can be shown that this problem is equivalent to a study of the stability of the equilibrium of another system similar to (1) [14].§ Therefore let us instead assume that our system as given is one which already has an equilibrium at $x = 0 \in G$, implying that

$$F(0, t) = 0 \qquad \forall\, t \geq t_0, \tag{2}$$

and it is the stability of this equilibrium which we wish to study. Note that (2) requires no actual knowledge of F but rather defines the origin of the state space E^n.

Later we will also consider the question of boundedness of motion when a small unknown forcing term is added to the above $F(x, t)$.

SYSTEM MODEL

Clearly the system model should be defined on the same state space E^n as the modeled system, and the set of possible initial states should also be the same. Therefore the system model should be described by a mathematical rule of the form

$$\dot{x} = f(x, t), \qquad x(t_0) = x_0 \in G \subset E^n, \qquad x \in E^n, \qquad t \geq t_0, \tag{3}$$

where f is a *known* $C^0 n$-vector function of the present state and time such that the model also has an equilibrium at the origin, i.e.

$$f(0, t) = 0 \qquad \forall\, t \geq t_0. \tag{4}$$

The system model is in error by an unknown n-vector function

$$g(x, t) \triangleq F(x, t) - f(x, t), \qquad x \in E^n, \qquad t \geq t_0 \tag{5}$$

such that

$$g(0, t) = 0 \qquad \forall\, t \geq t_0. \tag{6}$$

The modeled system (1) can therefore be written as

$$\dot{x} = f(x, t) + g(x, t), \qquad x(t_0) = x_0 \in G \subset E^n, \qquad x \in E^n, \qquad t \geq t_0. \tag{7}$$

† In effect we have adjoined a vector of position coordinates and a vector of velocity components to form a state vector x.

‡ This description of the physical system is in some ways a generalization, and in some ways a specialization, of the mathematical definition of a *dynamical system* [13].

§ An important and somewhat obscure point is that one must know what the given motion *is* before one can begin to examine its stability. An exception to this occurs for linear systems only.

Our objective is to obtain conditions such that stability statements valid for the equilibrium of the system model (3) are also valid for the modeled system (7). These conditions should not require explicit knowledge of $g(x, t)$.

STABILITY THEORY AND MODELING

The basic Liapunov definitions for stability of the equilibrium are found in Ref. [14].

Definition 1

The equilibrium is said to be *stable* if there exists for each $\varepsilon > 0$ a number $\delta > 0$ such that the inequality

$$|x_0| < \delta$$

implies

$$|x(t; x_0, t_0)| < \varepsilon \qquad \forall\, t \geq t_0.$$

Definition 2

The equilibrium is said to be *quasi-asymptotically stable* if there is a number $\delta_0 > 0$ such that from $|x_0| < \delta_0$ the relation

$$\lim_{t \to \infty} x(t; x_0, t_0)$$

follows.

Definition 3

The equilibrium is said to be *asymptotically stable* if it is both stable and quasi-asymptotically stable.

Definition 4

The equilibrium is said to be *unstable* if it is not stable.

Definition 5

The equilibrium is said to be *completely unstable* if there exists a number $\varepsilon > 0$ with the following property: after finite time, *each* motion $x(t; x_0, t_0)$ reaches the sphere $|x| = \varepsilon$, where $0 < |x_0| < \varepsilon$ and $t_1 \geq t_0$.

Note that these definitions involve arbitrary (but small) initial conditions. Obviously we do not want to consider initial conditions which are impossible, and this accounts for the introduction of the region G. Let us state this explicitly in the form of a general rule for modeling:

Rule 1. The model should be defined on the same state space E^n as the physical system and there should be an open region $G \subset E$ such that:

(a) $0 \in G$;

(b) every $x \in G$ is a possible initial state of the modeled system (7);

(c) every $x \in G$ is a possible initial state of the system model (3).

It is usually a violation of this rule which produces the infinite-sensitivity models found while studying the effect of small inertial-parameter changes [11].

We will also have a use for several other terms defined in Ref. [14].

Definition 6

The equilibrium is said to be *uniformly stable* if for each $\varepsilon > 0$ a number $\delta = \delta(\varepsilon) > 0$, depending only on ε, can be determined such that the inequality

$$|x(t; x_0, t_0)| < \varepsilon \qquad \forall\, t \geq t_0$$

follows from

$$|x_0| < \delta$$

for all $t_0 \geq 0$.

Definition 7

The equilibrium is called *uniformly asymptotically stable* if:

(i) the equilibrium is uniformly stable;

(ii) for every $\varepsilon > 0$ a number $\tau = \tau(\varepsilon)$ depending only on ε, but not on the initial instant t_0 can be determined such that the inequality

$$|x(t; x_0, t_0)| < \varepsilon \qquad t > t_0 + \tau$$

holds, provided x_0 belongs to a spherical domain R_η whose radius η is independent of ε.†

Definition 8

The equilibrium is called *exponentially stable* if there exist two positive constants α and β which are independent of the initial values x_0, t_0, such that for sufficiently small $|x_0|$ the inequality

$$|x(t; x_0, t_0)| < \beta |x_0| \, e^{-\alpha(t - t_0)}$$

is satisfied.

Definition 9

The equilibrium is called *exponentially unstable* if two positive constants α and β exist and if there are initial values (x_0, t_0) in every domain $R_{\eta,\tau}$ with arbitrarily small η and arbitrarily large τ such that

$$|x(t; x_0, t_0)| > B |x_0| \, e^{\alpha(t - t_0)}.$$

If this relation is valid for all initial points x_0 such that $|x_0|$ is sufficiently small, then the equilibrium is called *completely exponentially unstable*.‡

Definition 10

If the equilibrium is either exponentially stable or exponentially unstable, the motions $x(t; x_0, t_0)$ such that $|x_0|$ is sufficiently small are said to have *significant behavior*.

† $R_\eta = \{x \mid |x| \leq \eta\}$.

‡ $R_{\eta,\tau} = \{x, t) \mid |x| \geq \eta, t \geq \tau\}$.

Definition 11

If the equilibrium is either exponentially stable or completely exponentially unstable, the motions $x(t; x_0, t_0)$ such that $|x_0|$ is sufficiently small are said to have *intensive behavior*.

We now come to a key theorem [14]:

Theorem 1

Let the motions of (3) have intensive behavior and for sufficiently small $|x| < \delta$ let there be an estimate of the form

$$|g(x, t)| < b|x| \qquad \forall\, t \geq t_0$$

for the additional term of (7). Then, if b is sufficiently small, the equilibria of (3) and (7) have the same stability properties.

Hahn's conjecture

Hahn [14] makes the conjecture that "intensive" may be replaced by "significant" in this theorem. This would represent a considerably sharper result but it has not been proved as yet.

Theorem 1 is the type of statement we wish to be able to make. Essentially it says that if the model (3) has intensive behavior, then its equilibrium has exactly the same stability properties as the equilibrium of the modeled system provided the right-hand side of (3) is in error by a sufficiently small linear term and an arbitrary completely nonlinear term. That is, the model is a finite-sensitivity stability model. We therefore propose the following rule for stability models:

Rule 2. A stability model should predict intensive behavior.† Otherwise we generally can say nothing about the physical system stability *regardless* of how accurate the model may be.

As may be apparent it is often not an easy matter to determine whether or not a general model has intensive behavior. If, however, the model is linear and autonomous [$f(x, t) = Ax$, $A =$ const. $n \times n$ matrix) considerable simplification occurs. Definitions 3, 7, 8 become equivalent to one another and to the condition that all eigenvalues of A have negative real parts. Also Definition 5 and the last part of Definition 9 become equivalent to the condition that all eigenvalues of A have positive real parts. In this case Rule 2 becomes the following:

Rule 2′. A linear autonomous stability model should predict asymptotic stability or complete instability.

If Hahn's conjecture is correct, "complete instability" may be replaced by "exponential instability" in Rule 2′, implying only that at least one eigenvalue has a positive real part. Whether or not Hahn's conjecture is correct, there is one type of model which is definitely useless. This is a model whose equilibrium is stable, but not asymptotically stable. In this case it is well known [14] and easily shown by example that the stability properties of the physical system (7) are completely determined by the function $g(x, t)$, regardless of how "small" it may be, and $g(x, t)$ is of course unknown. In such a case Rule 2 becomes a design criterion which requires one to introduce a change (e.g. damping) in the physical system if one wishes to predict its behavior on the basis of a model.

† If Hahn's conjecture is correct, "intensive" may be replaced by "significant".

It is the violation of Rule 2 which produces the infinite-sensitivity models found while studying the effect of small damping-parameter changes [1–10].

CONSTANTLY-ACTING DISTURBANCES

We now consider the case where the modeled system (1) does *not* have an equilibrium at $x = 0$ due to some small unknown disturbing forces. Therefore we generally wish to be sure that for sufficiently small $|x_0|$ and sufficiently small disturbing forces the ensuing motions remain small. In this case the physical system is still described by

$$\dot{x} = f(x, t)+g(x, t), x(t_0) = x_0 \in G \subset E^n, \qquad x \in E^n, \qquad t \geq t_0. \tag{8}$$

where

$$f(0, t) = 0 \qquad \forall\, t \in [t_0, \infty) \tag{9}$$

but

$$g(0, t) \neq 0 \tag{10}$$

in general. The model is still given by (3). The desired behavior is described by the following definition [14]:

Definition 12

The equilibrium of (3) is called *totally stable*, if for every $\varepsilon > 0$ two positive numbers $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ can be found such that for every motion $x(t; x_0, t_0)$ of (8), the inequality

$$|x(t; x_0, t_0)| < \varepsilon \qquad \forall\, t \geq t_0$$

holds, provided that

$$|x_0| < \delta_1$$

and

$$|g(x, t)| < \delta_2 \qquad \forall\, (x, t) \in R_{\varepsilon, t_0}.$$

The following theorem guarantees this behavior [14]†:

Theorem 2

If the equilibrium of (3) is uniformly asymptotically stable, then it is also totally stable.

Using this theorem, we see that compliance with Rule 2 via exponential asymptotic stability will assure finite-sensitivity of the model (3) with respect to constantly acting perturbations; i.e. the model will be totally stable.‡ If the model (3) is linear and autonomous, $f(x, t) = Ax$, total stability is assured if all eigenvalues of A have negative real parts.

SYSTEMS WITH MEMORY

We will consider here physical systems having a memory of length $\tau > 0$, for which the model is the instantaneous system (3). To effect a comparison it is necessary to assume

† Hahn's proof of this theorem is in error. A corrected proof is given here in the Appendix.

‡ Exponential stability implies uniform asymptotic stability.

some mathematical form for this very general physical system. Many such forms are conceivable, but to be specific we will consider a physical system of the form

$$\dot{x} = f(x,t)+\int_{t-\tau}^{t} h(x(s),s)\,\mathrm{d}s \tag{11}$$

where the known function $f(x,t)$ satisfies $f(0,t) = 0$ for all $t \geq t_0$, and the unknown function h is bounded and continuous. Using the mean value theorem this can also be written as

$$\dot{x} = f(x,t)+\tau h(x(t-\eta(t)),t-\eta(t)) \tag{12}$$

where $0 < \eta(t) < \tau\,\forall\, t \geq t_0$.

By the same method used in the Appendix to prove Theorem 2, it is also possible to prove the following:

Theorem 3

If the equilibrium of (3) is uniformly asymptotically stable, then for every $\varepsilon > 0$ there exist two positive numbers $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ such that if:

(i) the initial condition function $x_0(\xi)$, $\xi \in (t_0-\tau, t_0]$, of (11) satisfies the bound

$$|x_0(t_0)| < \delta_1;$$

(ii) τ is small enough that

$$\tau|h(x,t)| < \delta_2 \qquad \forall\,(x,t)\,\varepsilon\, R_{\varepsilon,t_0-\tau}$$

then every motion of (11) satisfies the bound

$$|x(t;x_0(\xi),t_0)| < \varepsilon \qquad \forall\, t \geq t_0.$$

Again we see that the satisfaction of Rule 2 by means of exponential asymptotic stability of the equilibrium of the model (3) leads to bounded behavior of the motions of the physical system (11), provided these motions originate sufficiently near $x = 0$ and the memory length τ is sufficiently short.

If we were to fix τ and consider (12) to be a physical system involving a single variable time delay $\eta(t)$, we reach the same conclusion but interpret δ_2 as a bound on the magnitude of the delay function h, which was neglected in forming the model (3). Again satisfaction of Rule 2 by exponential asymptotic stability implies finite-sensitivity of the model.

There are many other ways in which memory in the physical system, not reflected in the instantaneous model, could appear. It is clearly impossible to consider them all, but it appears that Rule 2 should assure a finite-sensitivity model even though no general proof exists. Rule 2 is violated by the infinite-sensitivity model of Ref. [12].

EXAMPLES AND COMMENTS

Since extensive algebraic manipulations are required to discuss in detail even the simplest multi-degree-of-freedom nonconservative systems, let us note that even one-degree-of-freedom systems can serve to illustrate some aspects of the mathematical modeling problem. To demonstrate the need for Rule 1 let us consider a particle of extremely small mass, within a nonconservative force field such that the particle position x is exactly described by

$$m\ddot{x}-c\dot{x}-kx = 0, \qquad m > 0, \qquad c > 0, \qquad k > 0, \tag{13}$$

for every initial condition $(x_0, \dot{x}_0)$ in the region

$$G = \{(x_0, \dot{x}_0) | (x_0^2+\dot{x}_0^2) < \varepsilon\} \tag{14}$$

where ε is a given positive number. The physical system (13) has the eigenvalues $[c+\sqrt{(c^2+4mk)}]/2m > 0$ and $[c-\sqrt{(c^2+4mk)}]/2m < 0$ and the equilibrium $(x, \dot{x}) = (0, 0)$ is clearly unstable.

Now suppose m is much smaller than c or k, so much so that in modeling the system we overlook the inertial effect and write

$$c\dot{x}+kx = 0, \qquad c > 0, \qquad k > 0. \tag{15}$$

This system model has the single eigenvalue $-k/c$ and its equilibrium is exponentially asymptotically stable. Clearly this is not an acceptable system model and we note that Rule 1 is not satisfied. The set of possible initial states for the model is restricted to

$$\{(x_0, \dot{x}_0) | (x_0^2+\dot{x}_0^2) < \varepsilon, c\dot{x}_0+kx_0 = 0\} \tag{16}$$

and this is not the set G of (14); i.e. initial velocity and initial displacement are directly related for the model.

Although the inadequacy of the model is fairly apparent in the above example even without the use of Rule 1, less obvious but completely analogous problems arise in modeling the multi-degree-of-freedom case when the mass associated with a given degree-of-freedom is dynamically ignored and the degree-of-freedom itself is not ignorable kinematically.

To illustrate some of the implications of Rule 2, let us consider another particle in a force field such that the position x is exactly described by

$$\ddot{x}-\alpha_1\dot{x}+\alpha_2\dot{x}^3+x = 0, \qquad \alpha_1 > 0, \qquad \alpha_2 > 0 \tag{17}$$

where α_1 and α_2 are small compared to unity. The equilibrium $(x, \dot{x}) = (0, 0)$ is clearly unstable. Suppose that in modeling this system we overlook the small parameter α_1 and write

$$\ddot{x}+\alpha_2\dot{x}^3+x = 0, \qquad \alpha_2 > 0. \tag{18}$$

The equilibrium of this system model is asymptotically stable, but not exponentially asymptotically stable. This is clearly not an acceptable model and it is rejected by Rule 2 since the behavior is not intensive.

If in modeling the system (17) we should overlook α_2 rather than α_1, we have the system model

$$\ddot{x}-\alpha_1\dot{x}+x = 0, \qquad \alpha_1 > 0. \tag{19}$$

The equilibrium of this system model is completely exponentially unstable. It is an acceptable model and complies with Rules 1 and 2.

For linear multi-degree-of-freedom systems the violation of Rule 2 leads to somewhat more interesting anomalies. See for example Ref. [5] and compare the "model behavior" described in Fig. 2 with various other possible "physical behaviors" described in Figs. 4–7.

From the standpoint of modeling alone, the failure of a given model to satisfy Rules 1 and 2 implies only that we must seek a more accurate model which does satisfy these rules. This may not be even a theoretical possibility since there is no *a priori* assurance that the physical system itself satisfies Rule 2. However we have no other alternative if we wish to predict the actual stability properties of the physical system.

The designer, however, is in a happier position. To him the failure of his model to satisfy Rules 1 and 2 serves to indicate a strong possibility that the physical system will be an engineering failure, since small parametric changes, etc., are inevitable in a physical system and should not have disastrous consequences in a well-designed system. Therefore the failure of a fairly accurate model to satisfy Rules 1 and 2 should lead him to purposely introduce additional elements or effects into the physical system in such a way that the new model of the new physical system does in fact satisfy Rules 1 and 2.

CONCLUSIONS

We have considered the use of stability analysis of the equilibrium of an instantaneous system model as a tool for predicting the behavior of "near equilibrium" motions of a physical system. We have stated two rules and proved that infinite-sensitivity does not occur if the physical system is instantaneous and these rules hold. We have also given support to the belief that this is true for physical systems with memory. Additional support is given by the fact that all infinite-sensitivity examples known to the author violate one or both of these rules.

It is felt that these rules provide a reasonable basis for the analysis and design of non-conservative systems. If the design objective is to obtain a physical system such that small disturbances produce small responses, and such that all motions decay in the absence of disturbances, then these rules have already been employed for many years in the successful design of feedback control systems, although never explicitly stated in this form.

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APPENDIX

Proof of Theorem 2

Let (3) be uniformly asymptotically stable and let $f \in C^0$ in R_{η,t_0}. Then according to a theorem of Massera [14, 15] there exists a positive definite decresent function V whose total derivative for (3) is negative definite, and which has partial derivatives of any order. Therefore there exist three strictly monotonically increasing functions $\varphi(r)$, $\Psi(r)$ and $\chi(r)$ such that

$$\varphi(|x|) \leq V(x,t) \leq \Psi(|x|) \qquad (x,t) \in R_{h,t_0} \tag{20}$$

$$\dot{V}_{(3)}(x,t) \leq -\chi(|x|) \qquad (x,t) \in R_{h,t_0}. \tag{21}$$

Let ε be given and let $0 < \beta < \varphi(\varepsilon)$. Then there exists a number $\gamma = \gamma(\beta) > 0$ such that if $\tilde{x}$ satisfies $V(\tilde{x}, t) = \beta$ for any $t \geq t_0$, then $\gamma < |\tilde{x}| < \varepsilon$. Also

$$\dot{V}_{(3)}(x,t) < -\chi(\gamma) \qquad \forall\, x \ni \gamma < |x| < \varepsilon. \tag{22}$$

Now,

$$\dot{V}_{(8)}(x,t) = \dot{V}_{(3)}(x,t) + \nabla_x V(x,t) \,.\, g(x,t) \tag{23}$$

where $|g(x,t| < \delta_2$, $(x,t) \in R_{\varepsilon,t_0}$. Choosing δ_2 sufficiently small we then have by (22) and (23)

$$\dot{V}_{(8)}(x,t) < 0 \qquad \forall\, x \ni \gamma < |x| < \varepsilon. \tag{24}$$

Now choose $\delta_1 = \gamma$ and assume $|x_0| < \delta_1$. Therefore

$$V(x_0, t_0) < \beta \tag{25}$$

and (24) implies

$$V(x_{(8)}(t; x_0, t_0), t) < \beta \qquad \forall\, t \geq t_1, \qquad |x_0| < \delta_1. \tag{26}$$

Therefore,

$$|x_{(8)}(t; x_0, t_0)| < \varepsilon \qquad \forall\, t \geq t_0, \qquad |x_0| < \delta_1 \tag{27}$$

and Theorem 2 is proved.

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Абстракт—Большинство исследований указывает на то, что произвольно малые разницы между двумя неконсервативными динамическими системами могут быть результатом совершенно разных характеристик устойчивости этих двух систем. Это можно выяснить тем, что математическая модель является неясной при расчете и проектированию Физических неконсервативных систем. Используя основные результаты из теории устойчивости Ляпунова, предлагаются два принципа для математических моделей дискретных динамических систем, с целью избежения моделей с бесконечной чувствительностью. Исследуются некоторые общие типы погрешностей моделирования и представляется что эти принципы заключают конечно чувствительные модели.